Basic structures of the covariant canonical formalism for fields based on the De Donder–Weyl theory*

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Abstract

We discuss a field theoretical extension of the basic structures of classical analytical mechanics within the framework of the De Donder–Weyl (DW) covariant Hamiltonian formulation. The analogue of the symplectic form is argued to be the *polysymplectic* form of degree $(n+1)$, where n is the dimension of space-time, which defines a map between multivector fields or, more generally, graded derivation operators on exterior algebra, and forms of various degrees which play a role of dynamical variables. The Schouten-Nijenhuis bracket on multivector fields induces the graded analogue of the Poisson bracket on forms, which turns the exterior algebra of (horizontal) forms to a Gerstenhaber algebra. The equations of motion are written in terms of the Poisson bracket on forms and it is argued that the bracket with $H\widetilde{vol}$, where H is the DW Hamiltonian function and $\widetilde{vol}$ is the horizontal (i.e. space-time) volume form, is related to the operation of exterior differentiation of forms.

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1 Introduction

It has long been known that the Euler-Lagrange field equations may be written in the following covariant form reminiscent the Hamilton's form of equations of motion in mechanics:

$$\frac{\partial p_a^i}{\partial x^i} = -\frac{\partial H}{\partial y^a}, \qquad\qquad \frac{\partial y^a}{\partial x^i} = \frac{\partial H}{\partial p_a^i}. \tag{1}$$

Here $\{x^i\}$ $(i = 1,...,n)$ are space-time coordinates, $\{y^a\}$ $(a = 1,...,m)$ are field variables and the quantities p_a^i and H are given by the Lagrangean density $L = L(y^a, \partial_i y^a, x^i)$ as follows:

$$p_a^i \quad := \quad \frac{\partial L}{\partial(\partial_i y^a)}, \tag{2}$$

$$H \quad := \quad p_a^i \partial_i y^a - L. \tag{3}$$

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This formulation is known form the approaches of De Donder, Weyl and some others (see e.g. [1, 2] for a review and further references) to the variational calculus of multiple integrals. This, together with a close similarity to the Hamilton's equations in mechanics, is the reason why we call the formulation above the *De Donder–Weyl (DW) Hamiltonian formulation*. We shall also call n quantities p_a^i associated to each field variable y^a the *DW canonical momenta*, and the scalar density H is to be refered to as the *DW Hamiltonian function*.

The aforementioned Hamiltonian formulation of field equations has two major advantages over the standard Hamiltonian formalism in field theory – manifest covariance, in the sense that space and time variables enter the formulation on a completely equal footing, and finite dimensionality, in the sense that the analogue of the phase space, the space of variables (y^a, p_a^i, x^i) which we call the *DW phase space*, is finite dimensional. Because of the latter circumstances we may also call the Hamiltonian formulation above covariant and finite dimensional.

It is well known that the standard Hamiltonian formulation of equations of motion, both in mechanics and field theory, reveals the structures, like the Poisson brackets or the symplectic structure, which are important for transition to a quantum description of the dynamics along the lines of canonical or geometric quantization. The analogous structures in the DW Hamiltonian formulation are, essentially, not known, as well as the answer to the question whether it is possible to construct a quantum field theoretical formalism based on the finite dimensional canonical framework in field theory.

The latter problem was originally discussed by Born [3] and Weyl [4] in 1934. Then, certain interest to it arose again in the beginning of seventies, when a substantial progress was made in understanding the differential geometric structures behind the DW canonical theory for fields (see e.g. [5-11]), which culminated in the so-called multisymplectic formalism principally formulated in [9] (see [12-16] for recent developments). In this connection interesting ideas were suggested then by Guenther [17] and, more recently, Sardanashvily discussed possible applications of his closely related "multimomentum Hamiltonian formalism" to quantum field theory [18].

In this paper we concisely review our recent study of the canonical structure of the De Donder–Weyl Hamiltonian formulation in field theory [19, 20]. The aim of this research was to reveal those structures within the covariant finite dimensional canonical formalism for fields which generalize or are analogous to the structures in the formalism of classical mechanics which are known to be important for quantization. Compared to [19], Sect. 7 containes new results which will be presented in more details elsewhere.

2 A heuristic consideration

A construction of the canonical scheme corresponding to the DW Hamiltonian formulation faces a difficulty related to the fact that the number of field variables and the associated DW momenta is not equal, whereas the equality of the number of generalized coordinates and the conjugate momenta is an unavoidable prerequisite for the symplectic or the Poisson structure. Another problem is the suitable identification of the "evolution operation" generalizing the total time derivative in the canonical form of equations of motion in mechanics. In fact, in the left-hand side of the first of the DW Hamiltonian equations, eqs. (1), one has a space-time gradient while in the l.h.s of the second one there stands a space-time divergence. How to unify these two operations?

One can overcome these difficulties by noticing that the DW Hamiltonian equations may be written in terms of the form of degree $(n-1)$ associated with the DW momenta

$$p_a := p_a^i \partial_i \, \lrcorner \, \widetilde{vol}$$

in the following "unified" form

$$
\begin{aligned}
dy^a &= \text{"}\frac{\partial(H\widetilde{vol})}{\partial p_a}\text{"} := \frac{\partial H}{\partial p_a^i} dx^i, \\
dp_a &= -\frac{\partial(H\widetilde{vol})}{\partial y^a},
\end{aligned}
$$

where $\widetilde{vol} := dx^1 \wedge ... \wedge dx^n$ is the volume-form on the space-time manifold (for simplicity we use the coordinate systems with the unit metric determinant) and the expression in the quotation marks involves a formal differentiation of the n-form with respect to the $(n-1)$-form, which may be attributed certain sense with the help of the identity $\widetilde{vol}\, \delta_j^i = dx^i \wedge \partial_j \, \lrcorner \, \widetilde{vol}$. The formulation above suggests that the exterior differential may be a suitable generalization of the time derivative, and that the pair of variables (y^a, p_a), one of which is a 0-form and another one is a $(n-1)$-form, may be a suitable analogue of the canonically conjugate variables. The latter feature means that the Poisson bracket yet to be defined has to act on forms of various degrees and that the analogue of the canonical transformations may mix the forms of different degrees. All these properties are revealed in the formalism we present below.

3 Poincaré-Cartan form, classical extremals and the polysymplectic form

As a starting point of our construction we take the field theoretical analogue of the Poincaré–Cartan form which, being the fundamental object of the geometric approach to the calculus of variation, contains all the necessary information for constructing a canonical scheme. It is known that all the elements of the canonical formulation of mechanics may be recovered step by step from the Poincaré–Cartan form (see e.g. [21]). We present here a field theoretical generalization of these procedures.

The multidimensional or field theoretical analogue of the Poincaré–Cartan form written in terms of the DW Hamiltonian variables is the n–form (see e.g. [2,8,9,12-16,22])

$$\Theta = p_a^i dy^a \wedge \omega_i - H\omega, \tag{4}$$

where $\widetilde{vol} := dx^1 \wedge ... \wedge dx^n =: \omega$ and $\omega_i := \partial_i \, \lrcorner \, \widetilde{vol}$. The exterior differential of the Poincaré–Cartan form, the canonical $n+1$–form

$$\Omega_{DW} := d\Theta, \tag{5}$$

is of fundamental importance since the classical extremals are known to be its isotropic n–hypersurfaces in the $(m + mn + n)$–dimensional DW phase space with the coordinates $z^M := (y^a, p_a^i, x^i)$ ($M = 1, ..., m, m+1, ..., m+mn, ..., m+mn+n$). If we describe these hypersurfaces as the integral surfaces of the multivector field of degree n, $\overset{n}{X}$,

$$\overset{n}{X} := \tfrac{1}{n!} X^{M_1 ... M_n}(z) \, \partial_{M_1 \, ... \, M_n}, \tag{6}$$

where $\partial_{M_1 \, ... \, M_n} := \partial_{M_1} \wedge ... \wedge \partial_{M_n}$, which are defined as the solutions $z^M = z^M(x)$ of the equations

$$\overset{n}{X}{}^{M_1 ... M_n}(z) = \mathcal{N} \frac{\partial(z^{M_1}, ..., z^{M_n})}{\partial(x^1, ..., x^n)}, \tag{7}$$

with $\mathcal{N}$ depending on the parametrization of the surface, then the condition on $\overset{n}{X}$ to give the extremals is that it annihilates the canonical $(n+1)$ form, that is

$$\overset{n}{X} \lrcorner \, \Omega_{DW} = 0. \tag{8}$$

This condition specifies a part of the components of $\overset{n}{X}$ and using the parametrization (7) of its integral surfaces one arrives at the DW Hamiltonian equations, eqs. (1). One assumes that the DW extended phase space is a globally trivial bundle over the space-time (=horizontal subspace) with a typical fibre called a vertical subspace or a DW phase space with coordinates $z^v := (y^a, p_a^i)$ In these terms, eq. (8) specifies, in fact, only the vertical, $X^{v i_1 \, ... \, i_{n-1}}$, and a part of bivertical, $X^{a i \, i_1 \, ... \, i_{n-2}}_b$, components of $\overset{n}{X}$. However, the equation following from the bivertical components may be shown to be a consequence of

the DW Hamiltonian equations following from the vertical components. Thus, the vertical components of the canonical multivector field $\overset{n}{X}$ are sufficient for describing the classical dynamics.

Therefore, we introduce the notion of *vertical* multivector of degree p, $\overset{p}{X}{}^V$, having a component form

$$\overset{p}{X}{}^V := \frac{1}{(p-1)!} X^{v i_1 \ldots i_{p-1}} \partial_{v i_1 \ldots i_{p-1}}. \tag{9}$$

Then from (8) it follows that the DW canonical field equations may be recovered from the following condition on the vertical n-vector field:

$$\overset{n}{X}{}^V \lrcorner \, \Omega = (-)^n d^V H, \tag{10}$$

provided the parametrization in (7) is choosen in such a way that $\frac{1}{n!}\overset{n}{X}{}^{i_1 \ldots i_n} \partial_{i_1 \ldots i_n} \lrcorner \, \omega = 1$. Here d^V is the vertical exterior differential: $d^V \ldots := dz^v \wedge \partial_v \ldots$ and the form

$$\Omega := -dy^a \wedge dp_a^i \wedge \omega_i, \tag{11}$$

which is defined as the vertical exterior differential of the vertical (=non-horizontal) part of the Poincaré-Cartan form: $\Omega := d^V \Theta^V, \Theta^V := p_a^i dy^a \wedge \omega_i$, is to be refered to as the *polysymplectic* form.

4 Hamiltonian multivector fields and forms

Eq. (10) may be viewed as a map from 0–forms to vertical n–vectors. In general, the polysymplectic form maps horizontal p–forms $\overset{p}{F}$ ($p = 0, ..., n-1$), which are defined to have a form

$$\overset{p}{F} := \frac{1}{p!} F_{i_1 \ldots i_p}(z) dx^{i_1 \ldots i_p}, \tag{12}$$

where $dx^{i_1 \ldots i_p} := dx^{i_1} \wedge ... \wedge dx^{i_p}$, to vertical multivectors of degree $q = n - p$, $\overset{q}{X}_F$:

$$\overset{q}{X}_F \lrcorner \Omega = d^V \overset{p}{F}. \tag{13}$$

We call *Hamiltonian* the vertical multivector field fulfilling (13) for some horizontal form F. Similarly, the horizontal forms to which a vertical multivector can be associated through the map (13) are to be called the *Hamiltonian forms*. The multivector field $\overset{p}{X}_F$ is refered to as the Hamiltonian p–(multi)vector field associated with the form F.

Note that the map (13) given by the polysymplectic form has a non-trivial kernel formed by the Hamiltonian multivector fields $\overset{p}{X}_0$ fulfilling

$$\overset{p}{X}_0 \lrcorner \Omega = 0, \tag{14}$$

which are to be refered to as the *primitive* Hamiltonian fields. It is apparent that the Hamiltonian multivector field associated with a given form is determined up to an arbitrariness related to addition of a primitive field of the same degree. Therefore, the map (13) should be rather viewed as a map of forms to the equivalence classes of Hamiltonian multivector fields modulo addition of primitive fields. The quotient space of Hamiltonian multivector fields modulo primitive ones is the space on which the polysymplectic form may be considered as nondegenerate.

On the other hand, the very existence of the map from forms to multivector fields imposes restrictions on the forms themselves. These restrictions are implicit in the notion of Hamiltonian form. Let us consider an example of a form of degree $(n-1)$, $F := F^i \partial_i \lrcorner \, \widetilde{vol}$ to which a Hamiltonian

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vector field $X_F := X^a \partial_a + X^i_a \partial^a_i$ is associated by the map $X_F \,\lrcorner\, \Omega = d^V F$ or, in terms of components, $X^v \partial_v \,\lrcorner\, (-dy^a \wedge dp^i_a \wedge \omega_i) = (\partial_a F^i dy^a + \partial^a_j F^i dp^j_a) \wedge \omega_i$, whence it follows

$$X^i_a = \partial_a F^i, \tag{15}$$
$$-X^a \delta^i_j = \partial^a_j F^i. \tag{16}$$

It is clear that the latter equation imposes a restriction on an admissible dependence of the components of F on the DW momenta. Using the integrability condition of (16) one discloses that the most general admissible form of F^i compatible with (16) is

$$F^i(y,p,x) = f^a(y,x)p^i_a \omega_i + g^i(y,x). \tag{17}$$

5 Generalized canonical symmetry and the brackets of Hamiltonian multivector fields and forms

The hierarchy of maps (13) may be viewed as a local consequence (which hold globally for Hamiltonian multivector fields) of the following hierarchy of symmetries of the polysymplectic form

$$\pounds_{\overset{p}{X}} \Omega = 0 \quad (p = 1, ..., n) \tag{18}$$

which are expressed in terms of the generalized Lie derivatives with respect to the vertical multivector fields

$$\pounds_{\overset{p}{X}} \mu := \overset{p}{X} \,\lrcorner\, d^V \mu - (-1)^p d^V (\overset{p}{X} \,\lrcorner\, \mu). \tag{19}$$

The graded symmetry above extends in an apparent way the canonical symmetry of the symplectic form known from mechanics. We call the multivector fields fulfilling (18) *locally Hamiltonian*.

The notion of Lie derivative with respect to a multivector field gives rise to the bracket operation on locally Hamiltonian multivector fields:

$$[\overset{p}{X}_1 , \overset{q}{X}_2] \,\lrcorner\, \Omega := \pounds_{\overset{p}{X}_1} (\overset{q}{X}_2 \,\lrcorner\, \Omega). \tag{20}$$

It is easily revealed that

$$deg([\overset{p}{X}_1 , \overset{q}{X}_2]) = p + q - 1, \tag{21}$$
$$[\overset{p}{X}_1 , \overset{q}{X}_2] = -(-1)^{(p-1)(q-1)}[\overset{q}{X}_2, \overset{p}{X}_1], \tag{22}$$
$$(-1)^{g_1 g_3}[\overset{p}{X} , [\overset{q}{X} , \overset{r}{X}]] + (-1)^{g_1 g_2}[\overset{q}{X} , [\overset{r}{X} , \overset{p}{X}]]$$
$$+ (-1)^{g_2 g_3} [\overset{r}{X} , [\overset{p}{X} , \overset{q}{X}]] = 0, \tag{23}$$

where $g_1 = p - 1$, $g_2 = q - 1$ and $g_3 = r - 1$. Therefore, the bracket above is the Schouten-Nijenhuis (SN) bracket of vertical multivector fields [23]. Clearly, the set of locally Hamiltonian multivector fields is a graded Lie algebra with respect to the SN bracket.

The SN bracket of two Hamiltonian multivector fields gives rise to the Poisson bracket of the Hamiltonian forms they are associated with:

$$\begin{aligned}
[\overset{p}{X}_1 , \overset{q}{X}_2] \,\lrcorner\, \Omega &= \pounds_{\overset{p}{X}_1} d^V \overset{s}{F}_2 \\
&= (-1)^{p+1} d^V (\overset{p}{X}_1 \,\lrcorner\, d^V \overset{s}{F}_2) \\
&=: -d^V \{[\overset{r}{F}_1, \overset{s}{F}_2]\},
\end{aligned} \tag{24}$$

where $r = n - p$ and $s = n - q$. From the definition of the bracket of Hamiltonian forms it follows

$$\{[\overset{r}{F}_1, \overset{s}{F}_2]\} = (-1)^{(n-r)} X_1 \,\lrcorner\, d^V \overset{s}{F}_2 = (-1)^{(n-r)} X_1 \,\lrcorner\, X_2 \,\lrcorner\, \Omega. \tag{25}$$

One can easily prove the following properties of the Poisson bracket defined above

(i) degree

$$deg\{[\overset{r}{F}_1, \overset{s}{F}_2]\} = r + s - n + 1, \tag{26}$$

(ii) graded antisymmetry

$$\{[\overset{r}{F}_1, \overset{s}{F}_2]\} = -(-1)^\sigma \{[\overset{s}{F}_2, \overset{r}{F}_1]\}, \tag{27}$$

where $\sigma = (n - r - 1)(n - s - 1)$.

(iii) graded Leibniz rule

$$\{[\overset{p}{F}, \overset{q}{F} \wedge \overset{r}{F}]\} = \{[\overset{p}{F}, \overset{q}{F}]\} \wedge \overset{r}{F} + (-1)^{q(n-p-1)} \overset{q}{F} \wedge \{[\overset{p}{F}, \overset{r}{F}]\}, \tag{28}$$

(iv) graded Jacobi identity

$$(-1)^{g_1 g_3} \{[\overset{p}{F}, \{[\overset{q}{F}, \overset{r}{F}]\}]\} +$$
$$(-1)^{g_1 g_2} \{[\overset{q}{F}, \{[\overset{r}{F}, \overset{p}{F}]\}]\} + (-1)^{g_2 g_3} \{[\overset{r}{F}, \{[\overset{p}{F}, \overset{q}{F}]\}]\} = 0, \tag{29}$$

where $g_1 = n - p - 1$, $g_2 = n - q - 1$ and $g_3 = n - r - 1$. Therefore, the Poisson bracket defined in (24) equippes the set of Hamiltonian forms with the structure of a Gerstenhaber algebra [24].

6 The bracket form of equations of motion

The analogy with the Hamiltonian formalism in mechanics suggests that the bracket with the DW Hamiltonian function H is related to the equation of motion. For the bracket of a $(n-1)$-form $F := F^i \omega_i$ with H one has

$$\{[H, F]\} = X_F \lrcorner \, d^V H = X_F{}^a \partial_a H + X_F{}_a^i \partial_i^a H$$

where the components of X_F are given by (15) and (16). Using DW Hamiltonian equations, eqs. (1), and introducing the notion of the *total* exterior differential $\boldsymbol{d}$ of a horizontal form of degree p, $\overset{p}{F}$:

$$\boldsymbol{d}\overset{p}{F} := \partial_i z^M dx^i \wedge \partial_M \overset{p}{F} = \partial_i z^v dx^i \wedge \partial_v \overset{p}{F} + dx^i \wedge \partial_i \overset{p}{F} = \boldsymbol{d}^V F + d^{hor} F$$

one arrives at the following bracket representation of the equation of motion of a Hamiltonian form of degree $(n-1)$

$$*^{-1} \boldsymbol{d}F = \{[H, F]\} + \partial_i F^i, \tag{30}$$

where one makes use of the inverse Hodge duality operation: $*^{-1}\omega := 1$. The last term accounts for a possible explicit dependence of F^i on the space-time coordinates.

The generalization of the previous result to forms of arbitrary degree requires a certain extension of the canonical scheme developed above in sect. (4) and (5). In fact, because the bracket of any $\overset{p}{F}$ with H vanishes identically when $p < n - 1$ the only way out is to attribute certain meaning to the Poisson bracket with the DW Hamiltonian n-form $H\omega$. Therefore, we extend the hierarchy of maps (13) by including the n-forms

$$\tilde{X}_F \lrcorner \, \Omega = d^V \overset{n}{F}. \tag{31}$$

The formal degree of the object $\tilde{X}_F$ associated with an n-form is zero and one identifies it with the vertical vector-valued horizontal one-form $\tilde{X}$, $\tilde{X} := \tilde{X}^v{}_i \, dx^i \otimes \partial_v$, acting on Ω through the Frölicher-Nijenhuis inner product [25] (see also [26]) which is defined as follows

$$\tilde{X} \lrcorner \, \Omega := X^v{}_{\cdot k} dx^k \wedge (\partial_v \lrcorner \, \Omega). \tag{32}$$

Now, one can easily calculate the components of the vector-valued form $\tilde{X}_{H\omega} := \tilde{X}^v{}_i dx^i \otimes \partial_v$ associated with $H\omega$:

$$\tilde{X}^a{}_{\cdot k} = \partial_k^a H, \qquad \tilde{X}^i{}_{ak} \delta_i^k = -\partial_a H, \tag{33}$$

and reveal that the natural parametrization of $\tilde{X}_{H\omega}$:

$$\tilde{X}^v{}_{\cdot k} = \frac{\partial z^v}{\partial x^k}, \tag{34}$$

leads to the DW Hamiltonian field equations. It means that $\tilde{X}_{H\omega}$ also may be thought of as the analogue of the canonical Hamiltonian vector field and $H\omega$ as the analogue of the canonical Hamilton's function.

Now, formally generalizing (25) one defines the (tilded) bracket with $H\omega$ to be (cf. next section)

$$\{\![H\omega, \overset{p}{F}]\!\}^{\sim} := \tilde{X}_{H\omega} \,\lrcorner\, d^V \overset{p}{F}. \tag{35}$$

Calculating this bracket on extremals, that is using (1), one obtains for any p-form $\overset{p}{F}$

$$\boldsymbol{d}\overset{p}{F} = \{\![H\omega, \overset{p}{F}]\!\}^{\sim} + d^{hor} \overset{p}{F}. \tag{36}$$

This extends the bracket form of equations of motion to arbitrary forms and, in particular, reproduces the DW Hamiltonian field equations if suitable forms are substituted into the bracket. The tilde sign with a bracket means that it is calculated on extremals, so that the primitive fields cannot be factored out and, as a consequence, the graded antisymmetry of the bracket may be lacking. This is because the successive action of a multivector associated with $\overset{p}{F}$ and a vector-valued form associated with $H\omega$ has, in general, both graded commuting and graded anticommuting parts.

7 Non-Hamiltonian forms and graded derivation operators on differential forms

The analytical restriction imposed on Hamiltonian forms (cf. sect. (4)) excludes from the scheme constructed above some forms which are interesting from the point of view of the canonical theory under consideration. For example, the DW Hamiltonian n-form in the theory of interacting scalar fields y^a in Minkowski space-time, which is given by the Lagrangean density $L = \frac{1}{2}\partial_i y^a \partial^i y_a - V(y^a)$, may be written as $H\omega = -\frac{1}{2} * p^a \wedge p_a + V(y)\omega$ in terms of the $(n-1)$-form momenta variables $p_a = p_a^i \partial_i \,\lrcorner\, \widetilde{vol}$ which are Hamiltonian and their Hodge duals $*p_a := -p_a^i dx_i$ which are not. In passing to a quantum theory one will need to represent both p_a and $*p_a$ as certain operators on a Hilbert space and, therefore, one should know how to calculate the Poisson brackets with $*p_a$, which is impossible within the framework developed so far. To do this would require associating, through some extension of the map (13) given by the polysymplectic form, certain objects, more general than multivector fields, to the non-Hamiltonian forms, like $*p_a$ are. Such a generalization is an objective of this section. It is inspired by a recent work of Vinogradov [27] and further details will be published elsewhere.

Most generally, one can associate to a p-form $\overset{p}{F}$ the vertical graded derivation operator of degree $-(n-p)$

$$\overset{n-p}{D}_F \,\lrcorner\, \Omega = d^V \overset{p}{F} \tag{37}$$

The term vertical (graded) derivation means that $\overset{q}{D}$ lowers the vertical degree of a form on which it acts through the generalized inner product $\lrcorner$ (the sign which one omits in what follows for short) by one and the horizontal degree by $(q-1)$. Such a graded derivation operator may in principle be represented by means of the vertical $(q+p)$-vector-valued horizontal p-forms $\tilde{X} \in \Lambda_p^{q+p}$ acting

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on forms through the generalized Frölicher-Nijenhuis inner product (cf. [27]). Given the form $\overset{p}{F}$ there always exists some graded derivation of degree $(n-p)$ associated with it in the sense of (37), which can be properly represented in terms of the suitable multivector-valued form. Evidently, the graded derivation operator associated with a given form is defined modulo addition of primitive graded derivations $\overset{p}{D}_0$ $(p = 0, ..., n)$ annihilating the polysymplectic form:

$$\overset{p}{D}_0 \Omega = 0, \tag{38}$$

so that the symbol $\overset{n-p}{D}$ in (37) actually denotes the equivalence class of graded derivations of degree $(n-p)$ modulo primitive graded derivations of the same degree.

The simplest generalization of the formulae (25) for the Poisson brackets enables us to define the Poisson brackets of arbitrary forms as follows:

$$\{[\overset{p}{F}_1, \overset{q}{F}_2]\} := (-)^{(n-p)}\frac{1}{2}[\overset{n-p}{D}_1 \overset{n-q}{D}_2 + (-)^{(n-q)(n-p)}\overset{n-q}{D}_2 \overset{n-p}{D}_1]\Omega \tag{39}$$

$$= (-)^{(n-p)}\frac{1}{2}[\overset{n-p}{D}_1 d^V \overset{q}{F}_2 + (-)^{(n-q)(n-p)}\overset{n-q}{D}_2 d^V \overset{p}{F}_1]$$

This definition ensures the same graded antisymmetry of the bracket as in (27) and it reduces to (27) when both of the graded derivation operators are representable in terms of vertical multivectors. By a straightforward calculation one can also prove that the Poisson bracket above possesses all the properties of the Poisson bracket of Hamiltonian forms, eqs. (26)–(29). Therefore, the space of horizontal forms is a Gerstenhaber algebra with respect to the exterior product and the Poisson bracket defined in (39).

As an example, we shall calculate the graded derivation of degree $(n-1)$ associated with the non-Hamiltonian one-form $*p_a = -p_a^i dx_i$. We may represent this graded derivation (on the subspace of forms having a horizontal degree $(n-1)$) by a vertical n-vector-valued one-form

$$\tilde{X}_{*p_a} := \frac{1}{(n-1)!}\tilde{X}_{*p_a}{}^{v i_1 \ldots i_{n-1}}{}_k dx^k \otimes \partial_{v i_1 \ldots i_{n-1}}. \tag{40}$$

acting on forms through the generalized Frölicher-Nijenhuis inner product $\lrcorner$ (its definition in this case is apparent from eq. (42) below). It is given by

$$\tilde{X}_{*p_a} \lrcorner \Omega = d^V * p_a, \tag{41}$$

or in components

$$\tilde{X}_{*p_a} \lrcorner \Omega = \frac{1}{(n-1)!}\tilde{X}_{*p_a}{}^{v i_1 \ldots i_{n-1}}{}_k dx^k \wedge \partial_{v i_1 \ldots i_{n-1}} \lrcorner (dp_a^i \wedge dy^a \wedge \omega_i) \tag{42}$$

$$= n(-)^{c_{n-2}}\tilde{X}_{*p_a}{}^{v i_1 \ldots i_{n-1}}{}_k \epsilon_{i i_1 \ldots i_{n-1}} dx^k \wedge \partial_v \lrcorner (dp_a^i \wedge dy^a)$$

$$= -dp_a^i \wedge dx_i$$

where $c_p := 1 + 2 + ... + p$. Therefore,

$$n(-)^{c_{n-2}}\tilde{X}_{*p_a}{}^{v i_1 \ldots i_{n-1}}{}_k \epsilon_{i i_1 \ldots i_{n-1}} = -\delta_a^v g_{ik}. \tag{43}$$

Let us calculate now the bracket of $*p_a$ with an arbitrary $(n-1)$-form $F = F^i \omega_i$:

$$\{[F, *p_a]\} = -\frac{1}{2}[D_F \lrcorner d^V * p_a + (-)^{n-1}\tilde{X}_{*p_a} \lrcorner d^V F] \tag{44}$$

where the vertical graded derivation of degree -1 associated with F, D_F, is represented by a vertical bivector-valued one-form $\tilde{X}_F$,

$$\tilde{X}_F := \tilde{X}^{v i}{}_k dx^k \otimes (\partial_v \wedge \partial_i), \tag{45}$$

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acting on forms through the Frölicher-Nijenhuis-type inner product. From $\tilde{X}_F \lrcorner \, \Omega = d^V F$ written in components it follows

$$- \tilde{X}_{a\cdot k}^{ik} + \tilde{X}_{a\cdot k}^{ki} = \frac{1}{2} \partial_a F^i, \tag{46}$$

$$\tilde{X}_{\cdot\cdot k}^{ak} \delta_j^i - \tilde{X}_{\cdot\cdot j}^{ai} = \frac{1}{2} \partial_j^a F^i. \tag{47}$$

For calculating the bracket in (44) one needs the components $\tilde{X}_{aj}^i$ along the momenta directions only. Modulo addition of a primitive field $\tilde{X}_0 : \tilde{X}_0 \lrcorner \Omega = 0$, the solution of (46) reads

$$\tilde{X}_a^{ij}{}_k = -\frac{1}{2} \frac{1}{n-1} \delta_k^j \partial_a F^i \quad mod \, [\tilde{X}_0] \tag{48}$$

Now, note that the components of $\tilde{X}_F$ representing D_F are, essentially, given by the map (37) only on the subspace of forms of horizontal degree $(n-1)$. On the subspace of forms of horizontal degree p the operator D_F is rather represented by $\frac{n-1}{p} \tilde{X}_F$. This follows from the observation that the operator

$$\boldsymbol{I}_p := \frac{1}{p} (\delta_j^i dx^j \otimes \partial_i) \lrcorner, \qquad \boldsymbol{I}_0 := 1 \tag{49}$$

is the unit operator on the subspace of p-forms and also from the requirement that when the $(n-1)$-form F is Hamiltonian the graded derivation D_F should be equivalent to the derivation operator of degree -1 given by the vertical vector field X_F which is associated with a Hamiltonian F (cf. sect. (4)).

Taking these remarks into account, one obtains from (44)

$$\{\!\!\{ F, *p_a \}\!\!\} = \partial_a F^i dx_i. \tag{50}$$

It is interesting to reveal the following property

$$\{\!\!\{ *p_a, F \}\!\!\} = *\partial_a F^i \omega_i = *\{\!\!\{ p_a, F \}\!\!\} \tag{51}$$

which may be useful for representing the quantum operator corresponding to $*p_a$.

Note, that the simplicity of the example choosen above hides some tricky aspects of representation of the equivalence classes of graded derivations associated with forms in terms of the multivector-valued forms. They are to be treated elsewhere.

It is a great pleasure for me to contribute to the volume in honour of Professor D.D. Ivanenko. This is his group of general relativity at Moscow University where I started my way in theoretical physics ten years ago.

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